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Ermakov systems of arbitrary order and dimension: structure and linearization

W K Schief, C Rogers and A P Bassom†

School of Mathematics, The University of New South Wales, Sydney, NSW 2052, Australia

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Abstract. Ermakov systems of arbitrary order and dimension are constructed. These inherit an underlying linear structure based on that recently established for the classical Ermakov system. As an application, alignment of a $(2+1)$-dimensional Ermakov and integrable Ernst system is shown to produce a novel integrable hybrid of a $(2+1)$-dimensional sinh–Gordon system and of a conventional Ermakov system.

1. Introduction

The study of the coupled pair of nonlinear ordinary differential equations now known as Ermakov systems originated in 1880 [1]. In the intervening years, there has been extensive literature devoted to their analysis [2–30]. Ermakov systems arise, most notably, in nonlinear optics [31–35] but also in nonlinear elasticity [36, 37]. The main theoretical interest in such systems resides in the fact that they admit, generically, an integral of motion known as the Lewis–Ray–Reid invariant. In a recent key development, it was shown in [24] that the classical Ermakov system is, in fact, linearizable, that is, C-integrable in the terminology of Calogero [38–41]. It turns out that the Lewis–Ray–Reid invariant plays a crucial role in that linearization.

In a recent development [42], N-component Ermakov systems were introduced which can be iteratively reduced to $(N-2)$ linear equations augmented by a canonical two-component Ermakov system. Here, we extend the concept of Ermakov systems to higher dimension and order. The nonlinear systems so introduced admit a reduction to a linear base system which incorporates the higher-dimensionality and order, together with a canonical two-component Ermakov system.

An application to soliton theory is made through a particular reduction of a recently introduced $(2+1)$-dimensional Ernst system [43] which incorporates features of both an Ermakov and a $(2+1)$-dimensional sinh–Gordon system.

2. The classical Ermakov system

The classical Ermakov system as extended by Ray and Reid [6] adopts the form

$$\ddot{u} + \omega(t)u = \frac{\bar{f}(v/u)}{u^2 v} \qquad \ddot{v} + \omega(t)v = \frac{\bar{g}(u/v)}{v^2 u} \tag{1}$$

† Permanent address: Department of Mathematics, University of Exeter, Exeter EX4 4QE, UK.

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where $\bar{f}$, $\bar{g}$ and ω are arbitrary functions of their indicated arguments and the overdot designates the derivative $\mathrm{d}/\mathrm{d}t$. The associated first integral

$$I = \frac{w^2(u,v)}{2} + \int^{u/v} \bar{g}(\lambda)\,\mathrm{d}\lambda + \int^{v/u} \bar{f}(\mu)\,\mathrm{d}\mu \tag{2}$$

is termed the Lewis–Ray–Reid invariant of the system (1). Here, $w(u,v) = u\dot{v} - v\dot{u}$ denotes the Wronskian of u and v with respect to t. Relation (2) implies that w is a function of u/v only, whence (1) assumes the equivalent form

$$\ddot{u} + \omega(t)u = \frac{f(v/u)}{u^3} w^2(u,v) \qquad \ddot{v} + \omega(t)v = \frac{g(u/v)}{v^3} w^2(u,v). \tag{3}$$

It will prove convenient to adopt this formulation of the classical Ermakov system in the remainder of this paper.

On introduction of the change of dependent and independent variables

$$u = a(z)\phi \qquad v = b(z)\phi \qquad z = \psi/\phi \tag{4}$$

where ϕ and ψ are linearly independent solutions of the linear base equation

$$\ddot{\phi} + \omega(t)\phi = 0 \tag{5}$$

the Ermakov system (3) reduces to the autonomous form

$$a'' = \frac{f(b/a)}{a^3} w^2(a,b) \qquad b'' = \frac{g(a/b)}{b^3} w^2(a,b) \tag{6}$$

with the Wronskian $w(a,b) = ab' - ba'$ and the derivative $' = \mathrm{d}/\mathrm{d}z$.

The fundamental observation made by Athorne *et al* [24] is that the canonical system (6) is linearizable via a further change of dependent and independent variables. This reduction has been subsequently obtained by means of Lie group methods [44] and reinterpreted in [45]. Thus, in [45] it was shown that the reciprocal transformation

$$\tilde{a} = z/b \qquad \tilde{b} = 1/b \qquad \tilde{z} = a/b \tag{7}$$

decouples the system (6) into two *linear* equations of the *same* form, namely

$$\tilde{a}_{\tilde{z}\tilde{z}} - (\ln\mathcal{W})_{\tilde{z}}\tilde{a}_{\tilde{z}} + g(\tilde{z})\tilde{a} = 0 \qquad \tilde{b}_{\tilde{z}\tilde{z}} - (\ln\mathcal{W})_{\tilde{z}}\tilde{b}_{\tilde{z}} + g(\tilde{z})\tilde{b} = 0 \tag{8}$$

where

$$\mathcal{W} = \int^{\tilde{z}} [s^{-3} f(s^{-1}) - s g(s)]\,\mathrm{d}s. \tag{9}$$

This result was exploited in [42] to construct a Darboux transformation linking sequences of Ermakov systems. Here, it is the autonomizing transformation (4) which provides the guide to the construction of the generalized Ermakov systems to be introduced in the following section.

3. Generalized Ermakov systems

Here, we construct a natural generalization of Ermakov systems to arbitrary order and dimension. These nonlinear coupled systems admit reduction to a linear base equation encapsulating the order and dimension, together with the canonical Ermakov system (6) which captures the nonlinearity.

It is convenient, at the outset, to introduce a linear differential operator

$$L(\partial_i) \qquad \partial_i = \partial/\partial x^i \tag{10}$$

of arbitrary order and dimension. In analogy with (4) we set

$$u = a(z)\phi \qquad v = b(z)\phi \qquad z = \psi/\phi \tag{11}$$

where (a, b) is a solution of the canonical system (6) and ϕ, ψ are as yet unspecified functions of x^i. The following result is readily established.

Theorem. If (u, v) is connected to (ϕ, ψ) via the relations given by (11) where (a, b) is a solution of the canonical Ermakov system (6), then the identity

$$L(D_i)\begin{pmatrix} u \\ v \end{pmatrix} = QL(\partial_i)\begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{12}$$

is obtained, where the matrix-valued quantities D_i and Q are defined by

$$D_i = \partial_i + \frac{1}{w_i}\begin{pmatrix} vX_i & -uX_i \\ vY_i & -uY_i \end{pmatrix} \qquad Q = \begin{pmatrix} a - a'z & a' \\ b - b'z & b' \end{pmatrix} \tag{13}$$

with

$$X_i = \frac{f(v/u)}{u^3}w_i^2 \qquad Y_i = \frac{g(u/v)}{v^3}w_i^2 \tag{14}$$

and $w_i = uv_{x_i} - vu_{x_i}$.

Proof. The result is established by induction. Without loss of generality, we consider only operators of the form

$$L(\partial_i) = \partial_{i_1} \cdots \partial_{i_n}. \tag{15}$$

It is readily verified that†

$$[D_i, D_k] = 0 \tag{16}$$

and this, in turn, guarantees the compatibility of the Frobenius system

$$D_i P = 0. \tag{17}$$

In fact, it may be shown that

$$D_i Q = 0 \tag{18}$$

and, indeed, the general solution of (17) is given by

$$P = QC \tag{19}$$

where C is an arbitrary constant matrix.

If it is now assumed that

$$D_{i_1} \cdots D_{i_n}\begin{pmatrix} u \\ v \end{pmatrix} = Q\partial_{i_1} \cdots \partial_{i_n}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{20}$$

for specified n then

$$D_k L(D_i)\begin{pmatrix} u \\ v \end{pmatrix} = D_k QL(\partial_i)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = Q\partial_k L(\partial_i)\begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{21}$$

on use of (18). But, the initial condition

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q\begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{22}$$

holds identically so that the general result (12) follows by induction. □

† It should be pointed out that the validity of the notation $L(D_i)$ depends upon commutativity of the operators D_i.

The above theorem shows that, since in the generic case $\det Q = w(a, b) \neq 0$, the solution of the nonlinear coupled system

$$L(D_i)\begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{23}$$

for u and v admits the representation (11) where a and b are governed by the canonical Ermakov system (6) if and only if

$$L(\partial_i)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0. \tag{24}$$

Accordingly, in the above reduction, the higher dimensionality and order of the nonlinear system (23) are embodied in the linear base system (24), while its nonlinearity is encapsulated in the autonomous Ermakov system (6). The latter is linearizable, as indicated in section 2.

The classical Ermakov system is readily retrieved as a specialization of the above formulation. Thus, if we consider the system (23) associated with the operator $L(\partial_t) = \partial_t^n$ then its u-components corresponding to $n = 0, 1, 2, 3, \ldots$ yield, in turn,

$$\begin{aligned} &u = 0 \\ &u_t = 0 \\ &u_{tt} - X = 0 \\ &u_{ttt} - [(wX)_t + (vX - uY)X]w^{-1} = 0 \\ &\vdots \end{aligned} \tag{25}$$

where the index on X, Y and w has been dropped. The linear combination of $(25)_1$ and $(25)_3$ corresponding to the operator $L = \partial_t^2 + \omega(t)$ produces the classical Ermakov equation

$$u_{tt} + \omega(t)u = X. \tag{26}$$

Its companion

$$v_{tt} + \omega(t)v = Y \tag{27}$$

is obtained analogously via the v-component of (23).

In the above, the sequence (25) indicates that a hierarchy of Ermakov systems may be obtained by the action of D_t on their predecessors. Indeed, we may regard D_t and, more generally, D_i as recursion operators. This extension of the notion of Ermakov systems is examined in more detail in the next section.

4. Classical Ermakov hierarchies. A recurrence operator formalism

Recursion operators are well established in soliton theory as a tool for generating hierarchies of S-integrable nonlinear equations (see e.g. [46]). Likewise, C-integrable Burgers hierarchies have a natural recurrence operator formalism [47]. In the preceding, we have adopted the terminology of Calogero (see e.g. [41]) in which S-integrable and C-integrable systems are those amenable to some form of inverse scattering transform and linearization respectively.

In the following, we focus on Ermakov hierarchies of coupled ordinary differential equations with associated operators

$$L = \partial_t^n. \tag{28}$$

Thus, the Ermakov hierarchy

$$L(D_t)\begin{pmatrix} u \\ v \end{pmatrix} = \left[\partial_t + \frac{1}{w}\begin{pmatrix} vX & -uX \\ vY & -uY \end{pmatrix}\right]^n \begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{29}$$

where $w = uv_t - vu_t$ and

$$X = f(v/u)\frac{w^2}{u^3} \qquad Y = g(u/v)\frac{w^2}{v^3} \tag{30}$$

is considered. The relation

$$D_t^n \begin{pmatrix} u \\ v \end{pmatrix} = Q\partial_t^n \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{31}$$

which holds modulo the canonical Ermakov system (6) may be alternatively introduced via the vector-valued generating functions F and $\mathcal{F}$ given formally by

$$F = \sum_n \lambda^{-n} D_t^n \begin{pmatrix} u \\ v \end{pmatrix} \qquad \mathcal{F} = \sum_n \lambda^{-n}\partial_t^n \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{32}$$

where λ is a constant parameter. Thus, the relation (31) is encoded in the gauge transformation

$$F = Q\mathcal{F} \tag{33}$$

while the fact that D_t is a recurrence operator is represented by the relation

$$D_t F = \lambda F \tag{34}$$

or, explicitly,

$$F_t = \left[\frac{1}{w}\begin{pmatrix} -vX & uX \\ -vY & uY \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] F. \tag{35}$$

It is noted that, in view of (18), $D_t Q = 0$, so that Q is a solution of (34) with $\lambda = 0$. This suggests the notation $Q = F(0)$, whereupon (33) may be written as

$$F = F(0)\mathcal{F}. \tag{36}$$

Introduction of the further gauge transformation

$$\bar{F} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} F \tag{37}$$

where

$$\alpha = \exp \int^{v/u} s f(s)\,\mathrm{d}s \qquad \beta = \exp \int^{u/v} s g(s)\,\mathrm{d}s \tag{38}$$

reduces (35) to the canonical form

$$\bar{F}_t = \left[\begin{pmatrix} 0 & \bar{q} \\ \bar{r} & 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \bar{F} \tag{39}$$

with

$$\bar{q} = \frac{u\alpha}{w\beta} X \qquad \bar{r} = -\frac{v\beta}{w\alpha} Y. \tag{40}$$

The linear matrix equation (39) is reminiscent of the spatial part of the AKNS scheme [48] in soliton theory, namely

$$\Phi_x = \left[\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \Phi. \tag{41}$$

Under a gauge transformation akin to (36), namely

$$\Phi = \Phi(0)\Xi \tag{42}$$

where $\Phi(0)$ is a matrix-valued solution of (41) for vanishing spectral parameter λ, the scattering problem (41) becomes

$$\Xi_x = \lambda S \Xi \tag{43}$$

where

$$S = \Phi^{-1}(0)\sigma_3\Phi(0) \tag{44}$$

and $\sigma_3 = \mathrm{diag}(1, -1)$ is the usual Pauli matrix. The linear equation (43) in the new eigenfunction Ξ is the counterpart of $\mathcal{F}_t = \lambda\mathcal{F}$ in the case of the Ermakov hierarchy (29).

5. Application to a $(2+1)$-dimensional integrable Ernst-type equation

In a recent development, a strong $(2+1)$-dimensional integrable extension of the Ernst equation of general relativity [49]

$$\mathcal{E}_{z\bar{z}} + \frac{1}{2}\frac{\rho_{\bar{z}}}{\rho}\mathcal{E}_z + \frac{1}{2}\frac{\rho_z}{\rho}\mathcal{E}_{\bar{z}} = \frac{\mathcal{E}_z\mathcal{E}_{\bar{z}}}{\mathrm{Re}(\mathcal{E})} \qquad \rho_{z\bar{z}} = 0 \tag{45}$$

has been constructed in [43] via the LKR linear triad representation [50, 51]. In [43], various canonical reductions and specializations along with compatible Darboux-type transformations have been discussed. Amongst these is a $(2+1)$-dimensional sinh–Gordon system and analogues of a $(2+1)$-dimensional Darboux system descriptive of conjugate coordinate systems [52] and the well known self-induced transparency (SIT) equations. The $(2+1)$-dimensional Ernst-type equation has the form

$$\begin{aligned}
&\left[\partial_t - \frac{\mathcal{E}_t}{\mathrm{Re}(\mathcal{E})} + \mathrm{i}\,\mathrm{Re}(\rho)\right]\left[\mathcal{E}_{xx} + \mathcal{E}_{yy} - \frac{\mathcal{E}_x^2 + \mathcal{E}_y^2}{\mathrm{Re}(\mathcal{E})}\right] + \mathcal{E}_x\left[\frac{\mathcal{E}_t\bar{\mathcal{E}}_x - \bar{\mathcal{E}}_t\mathcal{E}_x}{2\,\mathrm{Re}(\mathcal{E})^2} + \mathrm{i}\rho_x\right]\\
&\qquad + \mathcal{E}_y\left[\frac{\mathcal{E}_t\bar{\mathcal{E}}_y - \bar{\mathcal{E}}_t\mathcal{E}_y}{2\,\mathrm{Re}(\mathcal{E})^2} + \mathrm{i}\rho_y\right] = 0\\
&\mathrm{i}(\rho_{xx} + \rho_{yy}) + \left[\frac{\mathcal{E}_t\bar{\mathcal{E}}_x - \bar{\mathcal{E}}_t\mathcal{E}_x}{2\,\mathrm{Re}(\mathcal{E})^2}\right]_x + \left[\frac{\mathcal{E}_t\bar{\mathcal{E}}_y - \bar{\mathcal{E}}_t\mathcal{E}_y}{2\,\mathrm{Re}(\mathcal{E})^2}\right]_y + \left[\frac{\mathcal{E}_x\bar{\mathcal{E}}_x + \mathcal{E}_y\bar{\mathcal{E}}_y}{2\,\mathrm{Re}(\mathcal{E})^2}\right]_t = 0
\end{aligned} \tag{46}$$

where $\mathcal{E}$ and ρ are complex functions. It is seen that in the t-independent case with $\mathrm{Im}(\rho) = 0$ the Ernst equation (45) is retrieved, where $z = x + \mathrm{i}y$.

It is readily shown that the ansatz

$$\mathcal{E} = a(\psi) \qquad \rho = \mathrm{i}\sigma \tag{47}$$

reduces the Ernst-type equation (46) to the autonomous complex equation

$$a'' = \frac{a'^2}{\mathrm{Re}(a)} \tag{48}$$

augmented by the real coupled system

$$\psi_{xxt} + \psi_{yyt} = \sigma_x\psi_x + \sigma_y\psi_y \qquad \sigma_{xx} + \sigma_{yy} = \tfrac{1}{2}c(\psi_x^2 + \psi_y^2)_t \tag{49}$$

where

$$\frac{a'\bar{a}'}{\mathrm{Re}(a)^2} = c \tag{50}$$

is a first integral of (48).

The coupled system (49) represents a disguised elliptic version of a recently proposed $(2+1)$-dimensional integrable extension of the classical sine–Gordon equation [50]. Thus, the change of variables

$$\tilde{\psi}_x = \psi_{ty}\cosh\psi - \sigma_y \sinh\psi \qquad \tilde{\psi}_y = -\psi_{tx}\cosh\psi + \sigma_x\sinh\psi \tag{51}$$

produces the sinh–Gordon system

$$\begin{aligned} &\left(\frac{\psi_{tx}}{\sinh\psi}\right)_x + \left(\frac{\psi_{ty}}{\sinh\psi}\right)_y + \frac{\psi_y\tilde{\psi}_x - \psi_x\tilde{\psi}_y}{\sinh^2\psi} = 0 \\ &\left(\frac{\tilde{\psi}_x}{\sinh\psi}\right)_x + \left(\frac{\tilde{\psi}_y}{\sinh\psi}\right)_y - \frac{\psi_y\psi_{tx} - \psi_x\psi_{ty}}{\sinh^2\psi} = 0 \end{aligned} \tag{52}$$

where we have set $c = 1$ without loss of generality. This system has been previously constructed by Schief [43] via an eigenfunction–adjoint eigenfunction constraint applied to the LKR integrable $(2+1)$-dimensional systems [53]. In fact, the subset of solutions to the Ernst-type equation (46) obtained above may be constructed via alignment with a suitable generalized Ermakov system. The first step in this procedure is to note that the linear terms in $(46)_1$ consist of

$$\mathcal{E}_{xxt} + \mathcal{E}_{yyt} + \mathrm{i}\,\mathrm{Re}(\rho)(\mathcal{E}_{xx} + \mathcal{E}_{yy}) + \mathrm{i}\rho_x\mathcal{E}_x + \mathrm{i}\rho_y\mathcal{E}_y. \tag{53}$$

Since the linear part of the generalized Ermakov system (23) is the same for u and v, we have to impose the constraint $\mathrm{Re}(\rho) = 0$ in order to identify (53) with the linear part of a generalized Ermakov system for $(u, v) = (\mathcal{E}, \bar{\mathcal{E}})$. This justifies the ansatz $(47)_2$. The latter, in turn, implies that the ρ-independent imaginary part of $(46)_2$ has to vanish identically. This is achieved via the ansatz $(47)_1$ or, more generally,

$$\bar{\mathcal{E}} = \bar{\mathcal{E}}(\mathcal{E}). \tag{54}$$

Moreover, since $\bar{\mathcal{E}}(\mathcal{E}) = \bar{a}(a)$, a combination of (48) and its complex conjugate produces the relation

$$\bar{\mathcal{E}}_{\mathcal{E}\mathcal{E}} + \frac{\bar{\mathcal{E}}_{\mathcal{E}}}{\mathrm{Re}(\mathcal{E})} = \frac{\bar{\mathcal{E}}_{\mathcal{E}}^2}{\mathrm{Re}(\mathcal{E})}. \tag{55}$$

Now, on introduction of the notation $(x^0, x^1, x^2) = (t, x, y)$ it is seen that the constraint (54) allows the definition of functions f and g according to

$$f(\bar{\mathcal{E}}/\mathcal{E}) = \frac{\mathcal{E}^3}{(\mathcal{E}\bar{\mathcal{E}}_{\mathcal{E}} - \bar{\mathcal{E}})^2\,\mathrm{Re}(\mathcal{E})} = \overline{g(\mathcal{E}/\bar{\mathcal{E}})} \tag{56}$$

whence we obtain the relations

$$\frac{\mathcal{E}_{x^i}^2}{\mathrm{Re}(\mathcal{E})} = \frac{f(\bar{\mathcal{E}}/\mathcal{E})}{\mathcal{E}^3} w_i^2 = X_i = \bar{Y}_i \tag{57}$$

in the notation of (14). On use of (55), (57) and the operators D_i as defined in (13), equation $(46)_1$ together with its complex conjugate then assume the compact form

$$[D_0D_1^2 + D_0D_2^2 - \sigma_{x^1}D_1 - \sigma_{x^2}D_2]\begin{pmatrix}\mathcal{E}\\ \bar{\mathcal{E}}\end{pmatrix} = 0 \tag{58}$$

which is nothing but the generalized Ermakov system (23) associated with the operator $L = \partial_t\partial_x^2 + \partial_t\partial_y^2 - \sigma_x\partial_x - \sigma_y\partial_y$. Accordingly, $(49)_1$ is identified as the linear base equation associated with the generalized Ermakov system (58). Furthermore, the autonomous equation (48) together with its complex conjugate constitutes the canonical system (6) for

$(a, b = \bar{a})$ with f and g defined by (56). In fact, (48) may be readily solved. Thus, bearing in mind its invariance under the Möbius transformation

$$a \to \frac{c_1 a + \mathrm{i}c_2}{\mathrm{i}c_3 a + c_4} \tag{59}$$

we may first assume that a is real and then boost the corresponding particular solution of (48) by the invariance (59) to obtain the general solution.

To summarize, it has been established that a suitably constrained $(2+1)$-dimensional integrable Ernst-type equation (46) incorporates the C-integrable generalized Ermakov system (58). The corresponding linear base equation turns out to be part of the reduced S-integrable system (49). This suggests a more general study of nonlinear systems which contain both S-integrable and C-integrable features.

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